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Quasi-coherent states and the spectral resolution of the q-Bose field operator

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Abstract. The single mode q-Bose field for $-1 \leq q < 1$ is shown to be a bounded operator with a continuous spectrum lying in the interval between $\pm(2/(1-q))^{1/2}$, thus becoming unbounded in the Bose limit ($q \to 1$). The generalized eigenstates are determined in terms of q-Hermite polynomials whose properties are exploited to obtain orthonormality relations. It is observed that these are connected to important results in combinatorics. It is shown that the eigenstates are products of *two* q-exponentials of the q-creation operator acting on the Fock vacuum, in contrast to ordinary coherent states which involve only a single q-exponential of the creation operator. A method of representing operators in normal-ordered form based on this representation is described. The spectral resolution is employed to compute the asymptotic temporal behaviour of the particle number expectation value in states produced by coupling the q-Bose field to a source. It is shown that the limit does not commute with the limit $q \to 1$. Specifically it is shown that for $0 \leq q < 1$ the asymptotic growth is linear in time in contrast to the quadratic growth for $q = 1$.

1. Introduction

It is known that in one or two space dimensions relativistic quantum field theory permits statistics other than Fermi and Bose, and this has been investigated for its intrinsic interest, in connection with Chern–Simons theories, and because of possible physical realization by 'anyons' [1–3]. Without the constraint of locality one may also consider the possibility of anomalous statistics arising in any number of space dimensions (in particular three) induced by deforming the canonical Heisenberg commutation relations of the creation and annihilation operators. Certain deformations of this type provide realizations of quantum groups [4, 5]. Deformations of the Heisenberg algebra also provide means of interpolating analytically between Fermi and Bose statistics [6]. To construct quantum mechanical systems with such 'intermediate statistics' we may employ the so-called 'q-mutation' relations defined as follows:

$$[a_j, a_k^\dagger]_q = \delta_{jk} \tag{1}$$

where j, k label degrees of freedom, and the bracket is defined by

$$[A, B]_q \equiv AB - qBA \tag{2}$$

with $q = 1$ for Bose and -1 for Fermi particles. In [7] I proved that this is a satisfactory space for quantum mechanics in that all vectors have positive norms.

In the present paper I wish to pursue the construction of quantum systems with intermediate statistics further, specifically to investigate some dynamical implications

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of intermediate statistics produced by the q-mutator deformation. Thus e.g. it is interesting to investigate the effect of small deviations from Fermi and Bose statistics which might conceivably be observed. There is extensive literature about the situation near $q=-1$, i.e. on experimental limits on possible violations of the Pauli principle by electrons [8]. Here we shall be concerned primarily with the bosonic side of the q interval. Thus, if one wishes to discuss experimental limits on deviation of the photon from Bose statistics, one must have a dynamical model for q near 1, and one must determine the observable effects of the deviation from unity.

In attempting to construct dynamical models for particles obeying q-mutator statistics for q near 1 it is natural to look for analogues of the creation of bosons by a source. Hence one is led to examine the analogue of the field operator, which, for a single mode, is

$$\phi_q \equiv (a+a^\dagger)/\sqrt{2}. \tag{3}$$

Here and in the following a and $a^\dagger$ will refer to the operators obeying the q-mutation relations. This paper provides some necessary tools for constructing dynamical models for q-bosons in that it elucidates the properties of the ϕ_q field. We will, however, only be concerned with a single mode, and will not confront the difficulties encountered in q-mutator theories with several modes.

We note that ϕ_q for $q<1$ differs from the case $q=1$ in several important ways. First of all its canonical conjugate is *not* the operator

$$\hat{\phi}_q \equiv (a-a^\dagger)/\mathrm{i}\sqrt{2} \tag{4}$$

which does not obey the canonical commutation relation with ϕ_q. We must, therefore, find an operator π_q which *is* the correct canonical conjugate. Moreover the number operator $\mathcal{N}$ is no longer $a^\dagger a$, i.e. the latter does not satisfy

$$[\mathcal{N}_q, a] = -a \qquad \text{and} \qquad [\mathcal{N}_q, a^\dagger] = a^\dagger. \tag{5}$$

Matrix elements of the number operator are of fundamental importance, and since the operator $\mathcal{N}$ will now be a complicated function of a and $a^\dagger$, some effort will be required to compute these.

Finally we note the most striking difference between ϕ_q for $q<1$ and ϕ_q for $q=1$. In the former case the field is *bounded* while in the latter it is *unbounded*. To see this observe that the eigenvalues of $a^\dagger a$ are

$$n_q = 1+q+q^2+\ldots+q^{n-1} = (1-q^n)/(1-q) \tag{6}$$

with $n=0, 1, \ldots$. Thus, for $q<1$, this is bounded by $(1-q)^{-1}$ and so also is the field operator ϕ_q. Note carefully that, in contrast, the particle number is *unbounded* for $q<1$ as it is for $q=1$.

We will see in section (2) that the spectrum of ϕ_q lies in the interval

$$-\lambda_0 \leqslant \lambda \leqslant \lambda_0 \tag{7}$$

in which

$$\lambda_0 \equiv (2/1-q))^{1/2}. \tag{8}$$

Moreover we shall see that the spectrum is continuous. The discrete spectrum is empty, so all of the eigenstates are generalized eigenstates, not proper eigenstates. In the following, however, we will often abuse language in the custom of physics and refer to them simply as eigenstates.

In the basis of its eigenfunctions, ϕ_q will be represented simply as multiplication by λ. Hence its conjugate π_q may formally be represented by $-\mathrm{i}\,\mathrm{d}/\mathrm{d}\lambda$. Although the derivative is not defined at the endpoints of the spectrum, the spectral measure vanishes at the endpoints, so that we may identify the two endpoints. Equivalently we need only consider a space of functions of λ which are periodic in λ with period equal to the width of the spectrum. This suggests that the formalism will be simplified if we introduce an angular variable θ. Thus instead of ϕ_q we will work with the related operator

$$\psi_q \equiv \cos^{-1}(\phi_q/\lambda_0) \tag{9}$$

with eigenstates belonging to eigenvalue θ denoted $|\theta\rangle$, and

$$0 \leqslant \theta \leqslant \pi. \tag{10}$$

The conjugate to ψ_q, denoted ρ_q, may then be represented by $-\mathrm{i}\, \mathrm{d}/\mathrm{d}\theta$ in the basis of θ wavefunctions.

When we examine the projections $\langle n|\theta\rangle$ of the field eigenstates on the Fock space states $|n\rangle$, we find that they are q-Hermite polynomials. These play an important role in combinatorics, and we see in section 3 that certain formulas of importance in combinatorics have simple interpretations in terms of the orthogonality and completeness of the field eigenstates.

Using the Fock space matrix elements we may also explore the most physically significant aspect of the boundedness of the field for $q<1$, i.e. its effect on the growth of the particle number expectation value $\langle\mathcal{N}\rangle$ in typical dynamical situations. Specifically we will discover that there is a non-commutativity in the limit of $\langle\mathcal{N}\rangle$ as $q\to 1$ and the limit of $\langle\mathcal{N}\rangle$ in a sequence of states for which it is becoming unbounded, e.g. in the states

$$|(t)\rangle \equiv \mathrm{e}^{-\mathrm{i}t\phi_q}|0\rangle \tag{11}$$

which correspond to the production of particles by a c-number source for $t\to\infty$. (Note that we absorb the parameter measuring the strength of the source into the time variable.) We shall show that the expectation value of the particle number in these states grows like t if $q<1$, but like t^2 if $q=1$.

In section 4 we will see that when the eigenstate $|\theta\rangle$ is represented in Fock space in terms of the action of powers of the creation operator $a^\dagger$ on the Fock vacuum, a surprisingly concise expression is obtained of the following form: Coherent states for $q=1$ are exponentials of $a^\dagger$ on the vacuum. For $q<1$ there is a natural q-deformation of the exponential referred to as the q-exponential, and there are states analogous to coherent states which are q-exponentials of $a^\dagger$ acting on the Fock vacuum. The eigenstates $|\theta\rangle$ are *not* of this form, but are *products of two* q-exponentials acting on the Fock vacuum. One cannot combine q-exponentials for $q<1$ by adding the exponents as one does for $q=1$, so that these do not simply reduce to q-coherent states. We therefore refer to them as *quasi-coherent* states. Like the coherent states, they provide a simple method for expressing operators in normal product form. Since such expressions are of importance in constructing quantum field theories, we indicate how normal ordering is done with the quasi-coherent state formalism. We will argue that the quasi-coherent state formalism is likely to be more useful for $q<1$ than the coherent state formalism.

2. Spectral resolution of the field and its conjugate

For $q=1$, the algebra of the creation and annihilation operators (Heisenberg algebra) can be realized by combinations of the operators x and $\mathrm{d}/\mathrm{d}x$ acting on a suitable space of functions. It was disocovered long ago by Rogers [9, 10] that much of ordinary analysis can be generalized by replacing the derivative operator d/dx with a 'q-derivative' defined by

$$\mathrm{d}_q f(x)=\frac{f(x)-f(qx)}{x(1-q)}. \tag{12}$$

All q-objects are obtained from $q=1$ objects by using d_q. Thus e.g.

$$\mathrm{d}_q x^n = n_q x^{n-1} \tag{13}$$

where n_q is the q-deformation of the integer n, namely

$$n_q = 1+q+\ldots+q^{n-1}. \tag{14}$$

The q-exponential is the solution of the equation

$$\mathrm{d}_q e_q(x) = e_q(x) \tag{15}$$

and is given by

$$e_q(x)=\sum_{n=0}^{\infty}\frac{x^n}{n_q!} \tag{16}$$

in which the q-factorial is

$$n_q! = n_q\cdot(n-1)_q\ldots 1. \tag{17}$$

All of the ingredients of analysis, e.g. the integral, and all of the basic functions, have q-analogous which play an important role in combinatorial analysis. To simplify expressions it is useful to introduce the standard symbol $(a;q)_n$ which is related to the q-factorial, and the q-exponential, namely:

$$(a;q)_n=(1-a)(1-qa)\ldots(1-q^{n-1}a). \tag{18}$$

Thus

$$(q;q)_n=(1-q)^n n_q! \tag{19}$$

and

$$(a;q)_\infty^{-1}=e_q(a/(1-q)). \tag{20}$$

The last relation is obtained by noting that (15) with the definition of d_q gives

$$e_q(x)=(1-(1-q)x)^{-1}e_q(qx). \tag{21}$$

This may be iterated and, with $q<1$, and the boundary value $e_q(0)=1$, yields

$$e_q(x)=\prod_{n=0}^{\infty}(1-xq^n(1-q))^{-1}=(x(1-q);q)_\infty. \tag{22}$$

Now we note that the algebra of creation and annihilation operators obeying the q-mutation relations can be realized as follows. Let $f(x)$ be a formal power series, and consider the action of the operators x and d_q on these functions. Evidently

$$[\mathrm{d}_q, x]_q = 1 \tag{23}$$

where the bracket is the q-mutator bracket. Moreover, d_q annihilates constants. Thus one can represent the q-mutator algebra on the space of formal power series by identifying a with d_q, $a^\dagger$ with x, and the vacuum with the number 1. No inner product has been defined, so d_q is not the adjoint of x. (The reader is cautioned that we are not representing a and $a^\dagger$ by the same linear combination of x and d_q as is done in the case $q=1$!) The point is that to compute eigenstates only the *algebraic* relationship between the a and $a^\dagger$ needs to be reproduced. Thus, suppose that a generalized eigenstate of ϕ_q belonging to the eigenvalue λ is expressed in the q-Fock space by

$$|\lambda\rangle = \sum_{n=0}^{\infty} a_n a^{\dagger n}|0\rangle. \tag{24}$$

Then if $f(x)$ is a formal power series with the *same* coefficients, i.e. if

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \tag{25}$$

it will have to satisfy the equation

$$(1\sqrt{2})(x+\mathrm{d}_q)f(x) = \lambda f(x) \tag{26}$$

and conversely.

Now from the definition of d_q one then has

$$f(x) = [1-\sqrt{2}\,\lambda(1-q)x+(1-q)^2x^2]^{-1}f(qx) \tag{27}$$

which may be iterated to produce

$$f(x) = \prod_{n=0}^{\infty} (1-\sqrt{2}\,\lambda(1-q)xq^n+(1-q)x^2q^{2n})^{-1}. \tag{28}$$

We may put this in a more familiar form by a change of variables. Let

$$\lambda(\theta) \equiv \lambda_0 \cos\theta \tag{29}$$

where

$$\lambda_0 \equiv (2/(1-q))^{1/2} \tag{30}$$

so that, in effect, we are determining the spectrum of the operator

$$\psi_q \equiv \cos^{-1}(\phi_q/\lambda_0). \tag{31}$$

Let $r=(1-q)^{1/2}x$, and $a_n=(1-q)^{n/2}b_n$. Then b_n is the coefficient of r^n in the power series expansion of the product

$$P_q(r, \cos\theta) = \prod_{n=0}^{\infty} (1-2rq^n\cos\theta+r^2q^{2n})^{-1}. \tag{32}$$

This product is a special case of the generating function for the ultra-spherical polynomials studied by Rogers [12]. The polynomials $H_n(\cos\theta|q)$ in $\cos\theta$ which it generates are called q-Hermite polynomials. Specifically one has the definition

$$\frac{1}{\Pi_{n=0}^{\infty}(1-2yrq^n+r^2q^{2n})} = \sum_{n=0}^{\infty}\frac{H_n(y|q)r^n}{(q;q)_n}. \tag{33}$$

It is known [12] that these functions satisfy the orthogonality relation

$$\int_{-1}^{1} H_m(y|q)H_n(y|q)w(y|q)\,\mathrm{d}y = \frac{2\pi(q;q)_n}{(q;q)_\infty}\delta_{mn} \tag{34}$$

in which

$$w(y|q)=(1-y^2)^{-1/2}\prod_{k=0}^{\infty}(1-2(2y^2-1)q^k+q^{2k}). \tag{35}$$

Thus if we define

$$|\theta\rangle\equiv\sum_{n}^{\infty}\langle n|\theta\rangle|n\rangle \tag{36}$$

where

$$|n\rangle=(n_q!)^{-1/2}a^{\dagger n}|0\rangle \tag{37}$$

are the normalized q-Fock states, and

$$\langle n|\theta\rangle\equiv(f(\theta)/(q;q)_n)^{1/2}H_n(\cos\theta|q) \tag{38}$$

with

$$f_q(\theta)=(q;q)_\infty|(\mathrm{e}^{2\mathrm{i}\theta};q)_\infty|^2/2\pi \tag{39}$$

we see that $|\theta\rangle$ will be an eigenstate of ϕ_q with eigenvalue $\lambda(\theta)$, and

$$\int_0^\pi\langle m|\theta\rangle\langle\theta|n\rangle\,\mathrm{d}\theta=\delta_{mn}. \tag{40}$$

Note that for convenience we have absorbed the function $f_q(\theta)$, which characterizes the spectral measure, into the definition of the states $|\theta\rangle$. For later reference we observe that it has a double zero at the boundaries of the spectrum, specifically

$$f_q(\theta)=a_q(\theta)\sin^2\theta \tag{41}$$

where

$$a_q(\theta)=(2/\pi)(q;q)_\infty|(q\,\mathrm{e}^{2\mathrm{i}\theta};q)_\infty|^2 \tag{42}$$

is non-vanishing on the entire spectrum.

Since the Fock-space states are complete, (40) shows that we have the desired resolution of unity in projectors of the eigenstates of ψ_q:

$$I=\int_0^\pi|\theta\rangle\langle\theta|\,\mathrm{d}\theta. \tag{43}$$

Thus the eigenvalues of ψ_q form a continuous spectrum with no discontinuities in the range $0\leqslant\theta\leqslant\pi$, i.e. the eigenvalues λ of the field ϕ_q are in

$$-\lambda_0\leqslant\lambda\leqslant\lambda_0. \tag{44}$$

As expected this becomes unbounded as $q\to1$.

Since wavefunctions in the theta basis are only defined on a finite λ set, $-\mathrm{i}\,\mathrm{d}/\mathrm{d}\lambda$ is not yet properly defined at the endpoints. Since the spectral measure f_q goes to zero at the endpoints, however, we can simply identify the endpoints, or equivalently restrict to periodic functions with the spectrum as period. Thus if ρ_q denotes the canonical conjugate to ψ_q, it is represented by $-\mathrm{i}\,\mathrm{d}/\mathrm{d}\theta$ in the θ basis. To distinguish its eigenstates from the Fock states, which are also labelled by integers, we denote them $|[k]\rangle$ so that

$$\langle\theta|[k]\rangle=(1/\sqrt{\pi})\,\mathrm{e}^{2\mathrm{i}k\theta}\qquad k=0,1,2,\ldots. \tag{45}$$

(Note that the factor 2 is required because the theta range is π rather than 2π.)

It is interesting to note the connection between simple quantum mechanical manipulations and formulae in q-analysis that are often quite laborious to derive. Thus the relation

$$\langle\theta|\theta'\rangle = \delta(\theta-\theta') \tag{46}$$

which results from (43) is connected to a formula satisfied by the q-Hermite functions which is a q generalization of the Mehler bilateral sum for the Hermite polynomials, namely [12]

$$\sum_{n=0}^{\infty} \frac{H_n(\cos\theta|q)H_n(\cos\phi|q)r^n}{(q;q)_n} = \frac{(r^2;q)_\infty}{|(r\,\mathrm{e}^{\mathrm{i}(\theta+\varphi)};q)_\infty(r\,\mathrm{e}^{\mathrm{i}(\theta-\varphi)};q)_\infty|^2} \qquad |r|<1. \tag{47}$$

Inserting (38) into (47), and using the completeness of the Fock basis, (47) becomes

$$\langle\theta|\theta'\rangle = \lim_{r\to 1^-} \frac{f_q(\theta)(r^2;q)_\infty}{|(r\,\mathrm{e}^{\mathrm{i}(\theta+\theta')};q)_\infty(r\,\mathrm{e}^{\mathrm{i}(\theta-\theta')};q)_\infty|^2}. \tag{48}$$

In passing to the limit one must be careful with the singularities. Let $u=\theta-\theta'$ and $v=\theta+\theta'$. There is a double pole at $u=0$ when $r\to 1$, but no singularity at $v=0$. The reason is that since θ, θ' are non-negative, the latter can occur only if both θ and θ' vanish. But in that case the function f_q has a strong enough zero to cancel the pole (see (41)). Thus, setting $r=\mathrm{e}^{-\varepsilon}$ one obtains the singular factor

$$\lim_{\varepsilon\to 0^+} (2\varepsilon/(\varepsilon^2+u^2)) = 2\pi\delta(u) \tag{49}$$

and (46) then appears.

This calculation suggests that standard quantum mechanical manipulations may be used to generate q-analysis identities which are quite tedious to derive directly. The q-Hermite function is an ingredient in the derivation by Rogers [10, 11] of the celebrated Rogers–Ramanujan identities which are essential in Baxter's solution of the hard hexagon model [10, 14]. q-analysis identities are also essential in the theory of partitions [11]. Equation (47) is itself known to have a combinatoric significance for $q=1$, and it is suspected that the q extension does also [12]. We shall pursue this idea elsewhere to obtain a quantum mechanical interpretation of the Rogers–Ramanujan identities.

3. Growth of particle number expectation values

Consider a sequence of states produced by coupling the field ϕ_q to a c-number current η. Thus let

$$|(t)\rangle = \mathrm{e}^{\mathrm{i}t\phi_q}|0\rangle = \mathrm{e}^{\mathrm{i}\tau\cos\psi_q}|0\rangle \tag{50}$$

with

$$\tau \equiv \lambda_0 t. \tag{51}$$

We should like to determine the behaviour of the particle number in such states as t becomes large. We can do this by computing one of the moments

$$N_p(t) \equiv \langle(t)|\mathcal{N}^p|(t)\rangle = \sum_{n=0}^{\infty} n^p|\langle n|(t)\rangle|^2 \tag{52}$$

for which we need to evaluate

$$\langle n|(t)\rangle = \int_0^\pi \mathrm{d}\theta (f_q(\theta))^{1/2} \langle n|\theta\rangle \, \mathrm{e}^{-\mathrm{i}\tau\cos\theta}. \tag{53}$$

One approach to this computation is to use the known [12] Fourier expansion of the H_q functions, namely

$$H_n(\cos\theta|q) = \sum_{j=0}^{n} \frac{(q,q)_n \cos((n-2j)\theta)}{(q,q)_j (q,q)_{n-j}}. \tag{54}$$

However, since we are primarily interested in the large η behaviour, we may avoid unnecessary tedium with the following observation: Note that one could do the calculation by first computing $\langle\theta|\mathcal{N}^p|\theta'\rangle$. Equation (47) appears to offer a way of accomplishing this. Thus one might formally differentiate with respect to r and then pass to the limit $r=1$ as we did to compute $\langle\theta|\theta'\rangle$. However, this is a very dangerous procedure and must be handled with great care because the distribution becomes very singular on test functions whose support includes the boundaries of the spectrum. Thus all interchanges of limits must be justified. However, this offers a clue to simplification for large t. We notice that at the boundary of the spectrum, where $\theta \approx 0$ or π, the singular factor is the same for all q with $0 \leqslant q < 1$. Basically this is because the order of the zero in the spectral measure near the boundaries (see (41)) is q independent.

Thus if, in the computation of some matrix element, the dominant contribution comes from the boundaries of the spectrum, one will find, in first approximation, that the answer is the same for all q in $0 \leqslant q < 1$. Now in computing asymptotic values of the expectation value of $\mathcal{N}$ in the state $|(t)\rangle$ for large t, there will be oscillations in the integrations over the spectrum which will damp out the contribution except at the *stationary points of the oscillating factor* $\exp(-\mathrm{i}\tau\cos\theta)$. *Since these occur at the* boundaries of the spectrum, we expect that the t dependence of the leading term in the asymptotic expansion will be the same for all q in $0 \leqslant q < 1$. Hence to determine this we need only study the very simple case $q=0$. Thus returning to (54) we notice that

$$H_n(\cos\theta|0) = \sin(n+1)\theta/\sin\theta \tag{55}$$

and the integral (53) is found to be a Bessel function, specifically

$$\langle n|(t)\rangle = \mathrm{i}^n (n+1)(\tau)^{-1} J_{n+1}(\tau). \tag{56}$$

We now select a moment to evaluate which is easy to do, namely

$$\langle (t)|(\mathcal{N}+1)^2|(t)\rangle = \tau^{-2} \sum_1^\infty n^4 (J_n(\tau))^2 = \tfrac{3}{8}(\tau)^2 + \tfrac{1}{2}. \tag{57}$$

This is a very interesting result. For $q=1$ one finds that the asymptotic behaviour of $\langle(\mathcal{N}+1)^2\rangle$ in the state $|(t)\rangle$ goes like t^4, rather than t^2. Thus $\langle\mathcal{N}\rangle$ will grow like t for $0 \leqslant q < 1$ instead of like t^2 as it does for $q=1$. The non-commutativity between the $q \to 1$ limit and the large particle number limit is clear from the divergence of the factor λ_0 of τ at $q=1$. The effect of first taking $q \to 1$ and afterwards $t \to \infty$ is evidently to replace the divergence in λ_0 with an extra power of t. This phenomenon is of such physical interest that I will pursue it at length in another paper.

4. Quasi-coherent states and normal ordering

Consider the way in which coherent states are used in field theory in the case $q=1$. Suppose an operator A is given in Fock representation by

$$A=\sum_{m,n} A_{mm}|m\rangle\langle n|=\sum_{m,n} A_{mm}(m!n!)^{-1/2}a^{\dagger m}|0\rangle\langle 0|a^n. \tag{58}$$

Then A can be obtained in normal order by using the fact that the vacuum projector can be written in the form

$$|0\rangle\langle 0|=:\mathrm{e}^{-a^\dagger a}:. \tag{59}$$

Hence

$$A=:\sum_{m,n} A_{mm}(m!n!)^{-1/2}a^{\dagger m}\,\mathrm{e}^{-a^\dagger a}a^n:. \tag{60}$$

This is not always the most convenient expression for normal ordering. Specifically, if we are quantizing a classical theory, the matrix elements of A are more simply expressed in the over-complete coherent state basis. Thus one obtains analogously:

$$A=:\int \mathrm{d}z\,\mathrm{d}z^*\,A(z,z^*)\,\mathrm{e}^{za^\dagger}\,\mathrm{e}^{-a^\dagger a}\,\mathrm{e}^{z^*a}: \tag{61}$$

in which we have absorbed the normalization of the coherent states into the function $A(z,z^*)$. The form of $A(z,z^*)$ is an object one takes from the classical theory, because the coherent states describe the minimum packet states which behave quasi-classically. In a system of bosons, for example, large z corresponds to macroscopically occupied states characterizing the classical field, and thus $A(z,z^*)$ can be inferred.

When we seek q-analogues of these manipulations we note first that there is an analogue of (59), namely

$$|0\rangle\langle 0|=:e_q^{-1}(a^\dagger a):. \tag{62}$$

which is proved in the appendix. Now the q analogue of the coherent state is $e_q(za^\dagger)|0\rangle$, and one can derive a q-analogue of the familiar over-completeness relation. Thus with (62) one can give a normal form representation of an operator in the q-Fock space identical in form to (61).

However, this q-normal form is not likely to be useful for $q<1$ for the following reason. The coherent state normal ordering was desirable for $q=1$ because the matrix elements of A between coherent states are related to classical behaviour when z is large. But as we saw in the last section, there are peculiar differences between the behaviour of objects like the number operator (which is an ingredient of the Hamiltonian) for $q<1$ and for $q=1$ so that one cannot expect the classical situation to provide a reasonable guide to quantization. We therefore must find a replacement for the q-analogue of (61).

We discover one by observing that the generating function (32) factorizes as

$$P_q(r,\cos\theta)=\prod_{n=0}^{\infty}|1-rq^n\,\mathrm{e}^{\mathrm{i}\theta}|^2=|(r\,\mathrm{e}^{\mathrm{i}\theta};q)_\infty|^2=|e_q(r\,\mathrm{e}^{\mathrm{i}\theta}/(1-q))|^2. \tag{63}$$

Hence we see that

$$|\theta\rangle=\mathscr{E}_q(\theta,a^\dagger)|0\rangle \tag{64}$$

where we have defined

$$\mathscr{E}_q(\theta, a^\dagger) = e_q(za^\dagger)e_q(z^*a^\dagger) \tag{65}$$

with

$$z = e^{i\theta}/(1-q)^{1/2}.$$

Thus, in contrast to the q-coherent states which are q-exponentials of the creation operator acting on the vacuum, here one has a product of *two* q-exponentials of the q creation operator which we call $\mathscr{E}_q$. The $\mathscr{E}_q$ exponential does not combine its exponents on multiplication, so this does not reduce to a single e_q. We therefore call these states *quasi-coherent states.*

Using this representation, the resolution of unity (43), and (62), we obtain a new way of normal ordering. Thus for any operator A we have

$$A = :\int_0^\pi \int_0^\pi \mathrm{d}\theta\, \mathrm{d}\theta' \langle\theta|A|\theta'\rangle \mathscr{E}_q(\theta, a^\dagger) e_q^{-1}(a^\dagger a) \mathscr{E}_q(-\theta',)a):. \tag{66}$$

If an operator, A, e.g. some Hamiltonian, is expressed in terms of the field, its matrix elements $\langle\theta|A|\theta'\rangle$ between eigenstates of the field, *which is all that is needed in* (66), *are immediately given.* Thus this provides a simple normal ordering technique which may be used in making a field theory for $q \leqslant 1$. The next task is to find the q-analogue of the functional integral, upon which work is in progress.

5. Conclusions

Having now understood the mathematical structure of the q-Bose field operator for a single mode, we are in a position to move one step forward in the difficult task of constructing a complete dynamical theory for particles obeying intermediate statistics. This task has two parts: first to construct a complete theory for a single mode, and next to extend this to a multimode theory. We have seen that, even for a single mode, the effect of the boundedness of the q-Bose field is manifest in peculiarities of behaviour for states with large particle number. Thus it is to be expected that some weak form of the Pauli principle will make its appearance when q deviates from unity. This is discussed by the author in another paper [15].

The problem of multimode fields is very difficult because, unlike in the q-deformations used in connection with quantum groups [4], the q-mutator relations (1) deform the Heisenberg relations for *distinct* modes as well as for identical modes. Some progress in developing techniques for the multimode problem have been reported recently by Greenberg [16].

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Appendix

Proof of the identity:

$$|0\rangle\langle 0| = :e_q^{-1}(a^\dagger a): \tag{A1}$$

First we show that

$$e_q^{-1}(x)=e_{1/q}(-x). \tag{A2}$$

To see this note that from the Leibnitz rule for d_q, namely [5]

$$\mathrm{d}_q(f(x)g(x))=f(qx)\,\mathrm{d}_q g(x)+(\mathrm{d}_q f(x))g(x) \tag{A3}$$

together with

$$\mathrm{d}_q(e_q^{-1}(x)e_q(x))=0 \tag{A4}$$

one obtains

$$\mathrm{d}_q(e_q^{-1}(x))=-e_q^{-1}(qx). \tag{A5}$$

But for any function

$$\mathrm{d}_q f(x)=\mathrm{d}_{\mathrm{q}/q} f(qx) \tag{A6}$$

so

$$\mathrm{d}_q e_{1/q}(-x)=\mathrm{d}_{1/q}e_{1/q}(-qx)=-e_{1/q}(-qx) \tag{A7}$$

which has the same form as (5).

Thus both sides of (2) have the same infinite product (see (21), (22)), and the same value at $x=0$, which proves (2).

Next one notes that

$$n_{1/q}=n_q/q^{n-1} \tag{A8}$$

so that

$$n_{1/q}!=n_q!/q^{n(n-1)/2}. \tag{A9}$$

Thus form (2)

$$e_q^{-1}(x)=\sum_{n=0}^{\infty}(-x)^n q^{n(n-1)/2}/n_q!. \tag{A10}$$

Finally observe that the q binomial coefficient satisfies the q-Pascal triangle identity

$$\binom{n}{m}_q=\binom{n-1}{m}_q+q^{n-m}\binom{n-1}{m-1}_q \tag{A11}$$

whence, by induction, there follows

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}_q q^{j(j-1)/2}=\delta_{n0}. \tag{A12}$$

But

$$\langle n|a^j a^{\dagger j}|m\rangle=\delta_{mn}n_q!/(n-j)_q! \tag{A13}$$

whence from (12)

$$\langle n|:e_q^{-1}(a^{\dagger}a):|m\rangle=\delta_{n0}\delta_{m0} \tag{A14}$$

which proves (A1).

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